The Exact Renormalization Group

Hidenori SONODA
Kobe University, Japan

14 November 2009 at the Cornwall Fest

Abstract

I explain how to formulate quantum field theory using the exact renormalization group (ERG).

My first impression of UCLA

was made by Mike one day in the early 80's.

I was a beginning graduate student at Caltech. Mike came to give a seminar...

Plan of the talk

1. ERG differential equation
2. Renormalization with ERG
3. Direct construction of the continuum limit
4. Realization of symmetry by ERG
5. Example: QED
6. Perspectives

Reference: Y. Igarashi, K. Itoh, H.S., Prog. Theo. Phys. Suppl. **181** (2009), 1

Motivation

1. Dimensional regularization is not good at handling chirality and supersymmetry.

2. Regularization with a momentum cutoff can handle chirality, but not necessarily gauge symmetry.

3. The exact renormalization group provides a systematic way of introducing a “smooth” momentum cutoff via the construction of a Wilson action.

4. It is important to understand how to incorporate symmetry (continuous, either global or local) into the Wilson action.

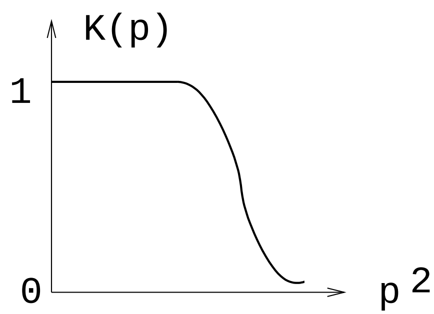
ERG differential equation

1. **Smooth momentum cutoff**

$$S_B[\phi] = -\frac{1}{2}\int_p \frac{p^2+m^2}{K\left(p/\Lambda_0\right)}\phi(-p)\phi(p) + S_{I,B}[\phi]$$

where $K(p)$ is a positive decreasing function of p^2, satisfying

(a) $K(p) = 1$ for $p^2 < 1$
(b) $K(p) \to 0$ as $p^2 \to \infty$



The propagator $\frac{K(p/\Lambda_0)}{p^2+m^2}$ decreases rapidly for $p^2 > \Lambda_0^2$.

2. **Decomposition** of the propagator ($\Lambda < \Lambda_0$)

$$\underbrace{\frac{K(p/\Lambda_0)}{p^2+m^2}}_{\phi} = \underbrace{\frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2+m^2}}_{\phi_h} + \underbrace{\frac{K(p/\Lambda)}{p^2+m^2}}_{\phi_l}$$

$$\phi = \phi_h + \phi_l$$

3. **The Wilson action** $S_\Lambda[\phi]$ is defined by

$$S_\Lambda[\phi_l] = -\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda)}\phi_l(-p)\phi_l(p) + S_{I,\Lambda}[\phi_l]$$

where

$$\begin{aligned} & \exp\left[S_{I,\Lambda}[\phi_l]\right] \\ \equiv \ & \int [d\phi_h] \exp\left[-\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0) - K(p/\Lambda)}\phi_h(-p)\phi_h(p) + S_{I,B}[\phi_l+\phi_h]\right] \end{aligned}$$

4. Alternatively, $S_{I,\Lambda}$ is defined by the **ERG differential equation** [Wilson & Kogut, 73; Wegner & Houghton, 73; Polchinski, 83]

$$-\Lambda\frac{\partial}{\partial\Lambda}S_{I,\Lambda} = \frac{1}{2}\int_p \frac{\Delta(p/\Lambda)}{p^2+m^2}\left[\frac{\delta S_{I,\Lambda}}{\delta\phi(-p)}\frac{\delta S_{I,\Lambda}}{\delta\phi(p)} + \frac{\delta^2 S_{I,\Lambda}}{\delta\phi(p)\delta\phi(-p)}\right]$$

where $\Delta(p) \equiv -2p^2 dK(p)/dp^2$.

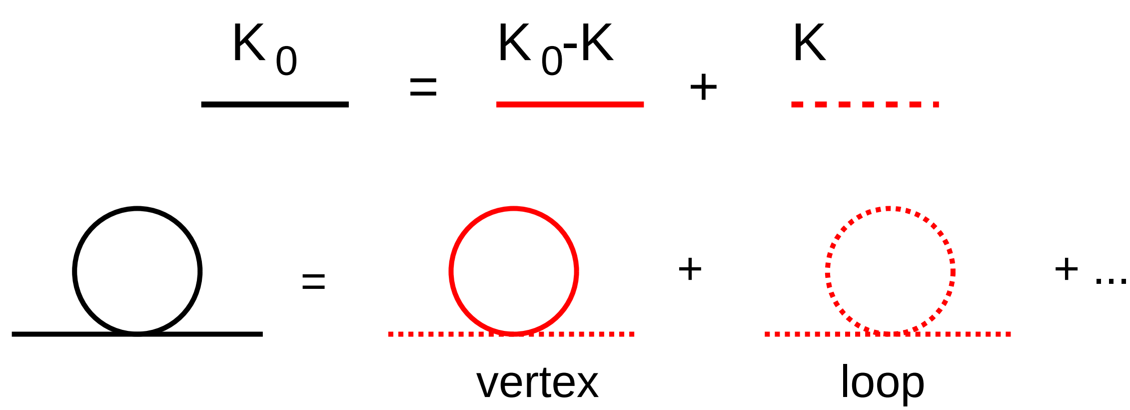
$S_{I,\Lambda}$ is determined by the differential equation and the initial condition

$$S_{I,\Lambda}\Big|_{\Lambda=\Lambda_0} = S_{I,B}$$

5. **No loss of information**

$$
\begin{aligned}
\langle \phi(p)\phi(-p)\rangle_{S_B} &= \frac{K(p/\Lambda_0)^2}{K(p/\Lambda)^2}\langle \phi(p)\phi(-p)\rangle_{S_\Lambda} \\
&\quad + K\,(p/\Lambda_0)\,\frac{1-K(p/\Lambda_0)/K(p/\Lambda)}{p^2+m^2} \\
\langle \phi(p_1)\cdots\phi(p_n)\rangle_{S_B} &= \prod_{i=1}^{n}\frac{K(p_i/\Lambda_0)}{K(p_i/\Lambda)}\cdot\langle \phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda}
\end{aligned}
$$

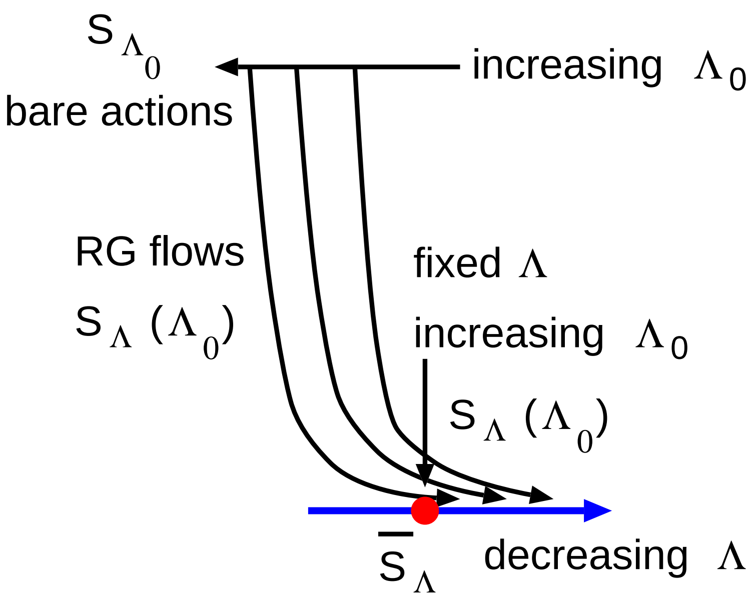
It's easy to understand this graphically.



6. Renormalization with ERG: parameterizing

$$S_{I,B} = -\int d^4x \left\{ \frac{z_2}{2} \partial_\mu \phi \partial_\mu \phi + \frac{\Delta m^2}{2} \phi^2 + \frac{\lambda_0}{4!} \phi^4 \right\}$$

we tune $z_2, \Delta m^2, \lambda_0$ so that $\lim_{\Lambda_0 \to \infty} S_\Lambda$ exists for any finite Λ.



We examine
$\bar{S}_\Lambda \equiv \lim_{\Lambda_0 \to \infty} S_\Lambda$
instead of
$\lim_{\Lambda_0 \to \infty} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_B}$

$\bar{S}_\Lambda$ gives the renormalized theory.

Direct construction of the continuum limit

1. The solution corresponding to the continuum limit has the low momentum expansion

$$\bar{S}_{I,\Lambda} \stackrel{p^2 \ll \Lambda^2}{\longrightarrow} \int d^4x \left[\left(\Lambda^2 a_2(\ln \Lambda/\mu) + m^2 b_2(\ln \Lambda/\mu) \right) \frac{1}{2}\phi^2 \right. \\ \left. + c_2(\ln \Lambda/\mu) \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + a_4(\ln \Lambda/\mu) \frac{1}{4!}\phi^4 \right]$$

$b_2(0), c_2(0), a_4(0)$ parameterize the solution. [Pernici et al., 98; H.S., 03]

For example, we can take

$$b_2(0) = c_2(0) = 0, \quad a_4(0) = -\lambda$$

2. The parameterization by $b_2(0), c_2(0), a_4(0)$ depends on the choice of an arbitrary scale μ.

3. The μ dependence of S_Λ is given by

$$-\mu \frac{\partial S_\Lambda}{\partial \mu} = \beta(\lambda)\, \mathcal{O}_\lambda + \beta_m(\lambda)\, m^2 \mathcal{O}_m + \gamma(\lambda)\, \mathcal{N}$$

where

$$\mathcal{O}_\lambda \equiv -\partial_\lambda S_\Lambda, \quad \mathcal{O}_m \equiv -\partial_{m^2} S_\Lambda + \cdots, \quad \mathcal{N} \equiv -\int_p \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} + \cdots$$

This gives the familiar (mass independent) RG equation

$$\left(-\mu\partial_\mu + \beta\partial_\lambda + \beta_m m^2 \partial_{m^2}\right) \langle \phi(p_1) \cdots \phi(p_n) \rangle = n\,\gamma\, \langle \phi(p_1) \cdots \phi(p_n) \rangle$$

[Pernici et al., 98; H.S., 06]

Realization of symmetry by ERG

1. S_Λ **with a finite** Λ **describes the continuum limit.**

2. **Any symmetry expected of the continuum limit must be present in** S_Λ**.**

3. S_Λ contains only the UV physics.

4. Integration over the low momentum modes is necessary to see how the symmetry manifests itself. (exact or broken)

5. **Ward-Takahashi identities**

$$\Sigma_\Lambda \equiv \int_p K\left(p/\Lambda\right) \left(\frac{\delta S_\Lambda}{\delta \phi(p)} \mathcal{O}_\Lambda(p) + \frac{\delta \mathcal{O}_\Lambda(p)}{\delta \phi(p)} \right) = 0$$

"quantum invariance" of the action under

$$\delta\phi(p) = K\left(p/\Lambda\right) \mathcal{O}_\Lambda(p)$$

- The second term is the jacobian.
- The correlation functions satisfy

$$\sum_{i=1}^{n} \langle \phi(p_1) \cdots \mathcal{O}(p_i) \cdots \phi(p_n) \rangle = 0$$

- We tune not only $S_{I,\Lambda}$ but also $\mathcal{O}_\Lambda$.

QED as an example

[Pernici et al., 98; H.S., 07]

1. the free action

$$S_{F,\Lambda} \equiv -\int_k \frac{1}{K(k/\Lambda)} A_\mu(-k) A_\nu(k) \left(k^2 \delta_{\mu\nu} - (1-1/\xi) k_\mu k_\nu \right) - \int_k \frac{1}{K(k/\Lambda)} \bar{c}(-k) c(k) k^2 - \int_p \frac{1}{K(p/\Lambda)} \bar{\psi}(-p) \left(\not{p} + im \right) \psi(p)$$

2. The interaction part of the action has the low momentum expansion

$$S_{I,\Lambda} \stackrel{p^2 \ll \Lambda^2}{\longrightarrow} \int d^4x \Big[\left(\Lambda^2 a_2(\ln \Lambda/\mu) + m^2 b_2(\ln \Lambda/\mu) \right) \frac{1}{2} A_\mu^2 + c_2(\ln \Lambda/\mu) \frac{1}{2} \partial_\mu A_\nu \partial_\mu A_\nu + d_2(\ln \Lambda/\mu) \frac{1}{2} (\partial_\mu A_\mu)^2$$

$$+a_4(\ln \Lambda/\mu)\frac{1}{8}(A_\mu^2)^2$$

$$+a_f(\ln \Lambda/\mu)\bar{\psi}\frac{1}{i}\partial\!\!\!/\psi + b_f(\ln \Lambda/\mu) im\bar{\psi}\psi$$

$$+a_3(\ln \Lambda/\mu) e\bar{\psi} A\!\!\!/\psi \Big]$$

3. 7 parameters $b_2(0), c_2(0), d_2(0), a_f(0), b_f(0), a_4(0), a_3(0)$

4. 3 fixed by normalization: $c_2(0) = a_f(0) = b_f(0) = 0$

5. 4 fixed by the WT identity: $b_2(0), d_2(0), a_4(0), a_3(0)$

$$\begin{cases} \delta A_\mu(k) &=& k_\mu \epsilon c(k) \\ \delta \bar{c}(-k) &=& \epsilon \frac{1}{\xi} k_\mu [A_\mu]_\Lambda(-k) \\ \delta\psi(p) &=& e\int_k \epsilon c(k)[\psi]_\Lambda(p-k) \\ \delta\bar{\psi}(-p) &=& -e\int_k \epsilon c(k)[\bar{\psi}]_\Lambda(-p-k) \end{cases}$$

where

$$[A_\mu]_\Lambda(-k) \equiv A_\mu(-k) + \frac{1-K(k/\Lambda)}{k^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{\delta S_{I,\Lambda}}{\delta A_\nu(k)} \overset{k^2 \ll \Lambda^2}{\longrightarrow} A_\mu(-k)$$

$$[\psi]_\Lambda(p) \equiv \psi(p) + \frac{1-K(p/\Lambda)}{p\cdot\gamma + im}\frac{\overrightarrow{\delta}}{\delta\bar\psi(-p)} S_{I,\Lambda} \overset{p^2 \ll \Lambda^2}{\longrightarrow} \psi(p)$$

6. **Gauge anomaly**

$$\Sigma_\Lambda \overset{p^2 \ll \Lambda^2}{\longrightarrow} \epsilon\int d^4x\,\frac{1}{i}\partial_\mu c\Big[\left(\Lambda^2\bar a_2 + m^2\bar b_2 - \bar c_2\partial^2\right)A_\mu - \bar a_3\bar\psi\gamma_\mu\psi + \bar a_4 A_\mu\frac{1}{2}A_\nu^2\Big]$$

where $\bar a_2, \bar b_2, \cdots$ are functions of $\ln\Lambda/\mu$.

We can tune $b_2(0), d_2(0), a_3(0), a_4(0)$ to make Σ_Λ vanish. $\rightarrow$ no anomaly

7. **Chiral anomaly** — for the massless chiral fermions, Σ_Λ gets an extra contribution proportional to

$$\int d^4x\, \epsilon_{\alpha\beta\gamma\delta} \partial_\alpha c \cdot \partial_\beta A_\gamma \cdot A_\delta$$

Perspectives

ERG is a viable alternative formulation of quantum field theory.

1. Short-term
   (a) construction of supersymmetric YM theories
   (b) elementary proof for the non-renormalization theorems
   (WZ model, sYM theories, chiral gauge theories, etc.)
   (c) perturbative construction of the Wilson-Fisher fixed point

2. Long-term
   (a) realization on a lattice — describing continuum physics using a lattice
   (b) ERG as a practical tool — higher order calculations?