SOME RECENT PROGRESS IN AdS/CFT

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Introduction

There are three maximally supersymmetric examples of AdS/CFT duality:

- **M2-brane Duality:** M theory on $AdS_4 \times S^7$ is dual to a SCFT in 3d. The supergroup is $OSp(8|4)$.
- **D3-brane Duality:** Type IIB superstring theory on $AdS_5 \times S^5$ is dual to a SCFT in 4d ($\mathcal{N} = 4$ SYM theory). The supergroup is $PSU(2,2|4)$.
- **M5-brane Duality:** M theory on $AdS_7 \times S^4$ is dual to a SCFT in 6d. The supergroup is $OSp(6,2|4)$.

The Type IIB / $\mathcal{N} = 4$ SYM Example

Gauge group $SU(N)$ corresponds to N units of flux ($\int_{S^5} F_5 \sim N$). The topological ('t Hooft) expansion of the gauge theory — large N and fixed λ, where

$$\lambda = g_{\text{YM}}^2 N,$$

corresponds to the loop expansion of the string theory. One also identifies

$$g_{\text{s}} \sim \frac{\lambda}{N} \quad \text{and} \quad R^2/\alpha' \sim \sqrt{\lambda}.$$

where R is the radius of the S^5 and the AdS_5.

The Type IIA / ABJM Example

One starts with M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, with N units of flux. This gives 3/4 maximal susy for $k > 2$.

The dual gauge theory is an $\mathcal{N} = 6$ superconformal Chern–Simons theory in 3d with gauge group $U(N)_k \times U(N)_{-k}$. The subscripts are the levels of the Chern–Simons terms. The ABJM theory also contains bifundamental scalar and spinor fields. This theory has a topological expansion with 't Hooft parameter

$$\lambda = N/k.$$

The orbifold $S^7/\mathbb{Z}_k$ can be described as a circle bundle over a CP^3 base. The circle has radius R/k, where R is the S^7 radius. When $k^5 \gg N$, there is a weakly coupled type IIA interpretation with

$$g_{\rm s} \sim (N/k^5)^{1/4}.$$

One then obtains the correspondences

$$R^2/\alpha' \sim \sqrt{\lambda} \quad \text{and} \quad g_{\rm s} \sim \lambda^{5/4}/N,$$

which is very similar to the other duality.

The metric of AdS_{p+2} in global coordinates is

$$ds^2[AdS_{p+2}] = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, ds^2[S^n].$$

In the case of two-point functions, the duality relates the energy E_A of a string state $|A\rangle$ (defined with respect to the global time coordinate t)

$$H_{\text{string}}|A\rangle = E_A|A\rangle,$$

to the conformal dimensions Δ_A of the corresponding gauge-invariant local operator $\mathcal{O}_A$ defined by

$$\langle\mathcal{O}_A(x)\mathcal{O}_B(y)\rangle \sim \frac{\delta_{AB}}{|x-y|^{2\Delta_A}}.$$

Specifically,

$$\Delta_A(\lambda, 1/N) = E_A(R^2/\alpha', g_{\rm s}).$$

The 't Hooft expansion of the dimension of $\mathcal{O}_A$ is

$$\Delta_A(\lambda, 1/N) = \Delta_A^{(0)} + \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{l=1}^{\infty} \lambda^l \Delta_{l,g}.$$

$\Delta_A^{(0)}$ is the classical (engineering) dimension, and the rest is called the anomalous dimension.

Almost all studies have focused on the planar approximation, $g = 0$, which is dual to free string theory. This may be fully tractable, but it is certainly not easy.

Approaches to testing the dualities

Method I. Identify tractable examples of classical solutions of the string world-sheet theory. Examine the spectrum of small excitations and compute their energies E_A. Identify the corresponding class of operators in the gauge theory and compute their dimensions Δ_A in the planar approximation. Then compare to E_A.

This requires an extrapolation from large λ, where the classical world sheet theory is valid, to small λ, where the gauge theory can be studied perturbatively.

Method II. Compare equations that determine E_A and Δ_A rather than the solutions. Approaches based on integrability and algebraic curves try to obtain equations of "Bethe type" on both sides and to match them.

It is much easier to study the world-sheet theory when the range of σ is infinite (rather than a circle). In the gauge theory analysis this corresponds to the thermodynamic limit of the Bethe equations arising from a spin-chain analysis.

Classical string solutions

The bosonic string action has 6 cyclic coordinates:

$$(t, \varphi_1, \varphi_2; \phi_1, \phi_2, \phi_3),$$

where the first 3 pertain to AdS_5 and the second 3 to S^5. These lead to conserved charges

$$(E, S_1, S_2; J_1, J_2, J_3).$$

One much-studied class of string solutions involves a line up the center of AdS_5, described by $\rho = 0$ and $t = \kappa\tau$, where κ is a constant and τ is the world-sheet time coordinate. These configurations have $S_1 = S_2 = 0$.

We parametrize S^5 as follows:

$$ds^2(S^5) = d\gamma^2 + \cos^2\gamma\, d\phi_3^2 + \sin^2\gamma\, ds^2(S^3)$$
$$ds^2(S^3) = d\psi^2 + \cos^2\psi\, d\phi_1^2 + \sin^2\psi\, d\phi_2^2.$$

The simplest solution is a point particle (collapsed string) encircling the sphere. This is described by

$$\gamma = \pi/2, \quad \phi_1 = \kappa\tau, \quad \psi = 0.$$

This has $J_2 = J_3 = 0$.

The quantum excitations of this solution have energies that can be expanded in powers of $1/J$ for large $J = J_1$, where

$$\kappa = J/\sqrt{\lambda}$$

is held fixed. This is equivalent to the BMN analysis of strings in a plane-wave background. One obtains

$$E - J \sim E_2(\kappa) + \frac{1}{J} E_4(\kappa) + \dots$$

The exact BMN result is

$$E_2 = \sum_{n=-\infty}^{\infty} \sqrt{n^2 + \kappa^2} N_n,$$

where $N_n = \sum_{i=1}^{8} \alpha_n^{i\dagger} \alpha_n^i +$ fermions, and the level matching condition is $\sum n N_n = 0$.

The assumption of BMN scaling predicts agreement with anomalous dimensions of operators in the dual gauge theory, even though one calculation is valid for large λ and the other for small λ. In fact, E_2 agrees perfectly, but agreement for E_4 breaks down at 3 loops.

Spinning string solutions

A generalization of the preceding involves circular or folded strings that are extended on the $S^3 \subset S^5$. These have $t = \kappa\tau$, $\rho = 0$, and $\gamma = \pi/2$, as before. But now one takes $\phi_1 = \omega_1 \tau, \quad \phi_2 = \omega_2 \tau, \quad \psi = \psi(\sigma)$.

The string equation of motion gives

$$\psi'' + \omega_{21}^2 \sin\psi \cos\psi = 0,$$

where $\omega_{21}^2 = \omega_2^2 - \omega_1^2$. This is a pendulum equation.

This equation has a first integral

$$\psi' = \omega_{21}\sqrt{q - \sin^2\psi}, \quad q = (\kappa^2 - \omega_1^2)/\omega_{21}^2.$$

The solution for $q < 1$, which involves the elliptic integrals $E(q)$ and $K(q)$, describes a folded string. The solution for $q > 1$, which involves the elliptic integrals $E(q^{-1})$ and $K(q^{-1})$, describes a circular string.

In the classical limit, the energy has the form

$$E = \sqrt{\lambda} F(J_1/\sqrt{\lambda}, J_2/\sqrt{\lambda}).$$

Dual gauge theory analysis

This string theory result can be extrapolated to small λ and compared to the dual gauge theory.

The operators that carry J_1, J_2 charges have the form

$$\mathcal{O}_{\alpha}^{J_1,J_2} = \mathrm{Tr}\left(Z^{J_1}W^{J_2}\right) + \ldots$$

where Z and W are complex scalar fields in the adjoint of $SU(N)$. Such a trace can be viewed as a ring configuration of an $S = 1/2$ quantum spin chain, where W corresponds to spin up and Z corresponds to spin down.

The dimensions are eigenvalues of the dilatation operator

$$\mathcal{D}\mathcal{O}_\alpha^{J_1,J_2}(x) = \sum_\beta D_{\alpha\beta}\mathcal{O}_\beta^{J_1,J_2}(x).$$

In the planar one-loop approximation one has a ferromagnetic Heisenberg spin chain, which is a well-known integrable system

$$\mathcal{H} = \sum_{i=1}^{J}\left(\frac{1}{4} - \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}\right).$$

This is solved using Bethe ansatz techniques. Higher orders have also been studied.

Strings spinning in AdS

The first example of this type is the straight folded string rotating in $AdS_3 \subset AdS_5$ (GKP) for large S. They found

$$E = 2\Gamma(\lambda) \log S + O(S^0),$$

where

$$\Gamma(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + O(\lambda^0) \quad \text{for} \quad \lambda \gg 1.$$

The dual gauge theory operators are

$$\mathrm{Tr}(D_+^{s_1} Z \, D_+^{s_2} Z) \quad s_1 + s_2 = S.$$

Their anomalous dimensions take the same form with

$$\Gamma(\lambda) = \frac{\lambda}{4\pi^2} + O(\lambda^2) \quad \text{for} \quad \lambda \ll 1.$$

An exact formula for the *cusp anomalous dimension* $\Gamma(\lambda)$ has been proposed (BES). The generalization of this duality for twist J operators

$$\mathrm{Tr}(D_+^{s_1} Z\, D_+^{s_2} Z \ldots D_+^{s_J} Z) \quad \sum s_l = S.$$

has been explored by Dorey, which he analyzes using an $SL(2)$ spin chain model. He finds that for large S these correspond to *spiky strings*.

Conclusion

There has been a lot of progress in testing AdS/CFT both on the string theory and the gauge theory sides exploiting the integrability of the respective theories.

I have described classical string solutions and the dual gauge theory operators. There has also been very interesting work exploring analogous constructions for the ABJM theory.

Happy Birthday Mike