The Linear Theory of Tearing Modes in periodic, cylindrical plasmas

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Resistive MHD

- \( \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \) (no energy principle)
- Role of resistivity
  - No frozen flux, \( \mathbf{B} \) can tear
  - If fluid moves relative to \( \mathbf{B} \) then

\[
\eta \mathbf{J} = \mathbf{v} \times \mathbf{B} \\
\mathbf{J} = \frac{\mathbf{v} \times \mathbf{B}}{\eta} \\
\mathbf{F} = \mathbf{J} \times \mathbf{B} = \frac{(\mathbf{v} \times \mathbf{B}) \times \mathbf{B}}{\eta} \\
= -\frac{B^2}{\eta} \mathbf{v} \quad \text{Restoring force}
\]

Vanishingly small \( \eta \) can allow tearing when \( B \to 0! \)
Resistive MHD

- Same energy source as in ideal MHD, but more states are accessible

- Dimensionless measure of resistivity

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times \mathbf{v} \times \mathbf{B} - \nabla \times \eta \frac{(\nabla \times \mathbf{B})}{\mu_0}
\]

\[
\text{LHS resistive term} \sim \frac{B/\tau_A}{\eta B/a^2 \mu_0} = \frac{\mu_0 a^2 / \eta}{\tau_A} = \frac{\tau_R}{\tau_A}
\]

\[
S = \frac{\tau_R}{\tau_A} = 6.5 \times 10^7 \frac{B_\theta, \text{Tesla}}{T_{e, \text{keV}} a_m} \frac{T_{e, \text{keV}}^{3/2}}{n_{20}^{1/2}}, \text{ and } \tau_R \equiv \frac{\mu_0 a^2}{\eta} \approx 50 \frac{T_{e, \text{keV}}^{3/2}}{Z_{\text{eff}}} \text{ sec}
\]

- $S=10^5-10^8$; although $S\gg1$, its finiteness permits topological changes and new instabilities
Experimental Importance

- Disruptions
- beta limits
- Stochastic magnetic fields
- Realization of reconnection phenomena
Consider torus with helical field

\[ \mathbf{B} = B_\phi \hat{\phi} + B_\theta \hat{\theta} \]

Field lines lie on circles
Magnetic Surfaces

Ideally, concentric tori

In general, there is shear
Add a small perturbation in radial magnetic field: $b_r$

with $\frac{b_r}{B} \ll 1$

Possible sources of perturbations:
- instability
- magnetic field error
- deliberate additional field

How can a small perturbation have a large effect on the field structure?
If $k_\parallel = 0$ then small magnetic fluctuation gives large field line excursion
add a perturbation (e.g., an instability)

\[
B = B(r) + b_r(r) \sin(m\theta - n\phi) \hat{r}
\]

Thus, wavenumber

\[
k = \frac{m}{r} \hat{\theta} - \frac{n}{R} \hat{\phi}
\]

consider region near \( k_\parallel = 0 \)

where

\[
k \cdot B = \frac{m}{r} B_\theta - \frac{n}{R} B_\phi = 0
\]

or where

\[
q(r) = \frac{m}{n} \quad \text{where} \quad q(r) = \frac{r B_\phi}{RB_\theta}
\]
Define a coordinate $\chi_\perp$, perpendicular to $\mathbf{B}$, $\hat{r}$: $\chi_\perp = m \theta - n \phi$
so that $B_r = b_r(r) \sin \chi_\perp$

Perturbation is constant along $\mathbf{B}$ at one radius
Field line equation

\[
\frac{dr}{B_r} = \frac{h_\chi d\chi_\perp}{B_\perp} \\
\frac{dr}{b_r(r) \sin \chi_\perp} = \frac{h_\chi d\chi_\perp}{B'_\perp r} \\
r dr = \frac{h_\chi}{B'_\perp} b_r \sin \chi_\perp \\
\sqrt{\frac{h_\chi}{4} \frac{b_r}{B'_\perp} \cos \chi_\perp} + C
\]
Field lines without perturbations

\[ r = \text{constant} \]
Field lines with perturbation

\[ r = \pm \sqrt{\frac{h \chi}{4} \frac{b_r}{B'_\perp}} \cos \chi_\perp + C \]

Reconnection has occurred:

\[ w = 2 \sqrt{\frac{h \chi b_r}{B'_\perp}} \]
Field lines respond to resonant perturbation in torus

\( n=1, m=3 \) perturbation:

\[ b_r = b \cos(3\theta - \phi) \]
SXR imaging of $m=3$ islands in tokamak

- Heating Power (NBI)
- Dalpa Divertor
- Beta Normalized
- $\text{(3,2)} + \text{(2,2)}$ Mirnov Activity
- TauE relative to ITER89P

- (2,2) SXR $t = 2.148 \text{ s}$
- (3,2) Mirnov $t = 2.16 \text{ s}$
SXR imaging of flux surfaces

With one dominant tearing instability

From a reversed field pinch experiment
(similar to a tokamak)
SXR tomography with 2 tearing modes
Disruptions can be due to overlapping islands

Asdex-U tokamak
References


“Tokamaks, 3rd ed.” Wesson

Cylindrical Tearing Mode: Linear Theory

1. Solve for magnetic perturbation in outer region:
   - use force balance, ignoring inertia
   - Use ideal MHD, ignoring resistivity

2. Outer solution determines stability, drives reconnection

3. Inner solution solves resistive MHD equations for narrow “current sheet”
   - Inertia, resistivity both important

4. Matching inner solution to step in \( B_r' \) determines growth rate
Outer region

Ion Equation of motion

$$\rho \frac{dV}{dt} = J \times B - \nabla P$$

in the outer region, inertia is negligible, and plasma is in force equilibrium

$$\nabla P = J \times B$$

and since the curl of a gradient is zero

$$\nabla \times J \times B = 0$$

Using $\nabla \cdot B = 0$ and $\nabla \cdot J = 0$

$$B \cdot \nabla J - J \cdot \nabla B = 0$$
Reduced MHD Ordering

The large aspect-ratio, reduced MHD ordering will be used. Taking $\epsilon = a/R \ll 1$, the equilibrium quantities are ordered as

$$B_\theta \sim \frac{1}{r} \frac{dB_\phi}{dr} \sim \epsilon B_\phi,$$

and $J_\theta \sim \epsilon J_\phi$

while the perturbed quantities are ordered as

$$b_\phi \sim \epsilon b_r \sim \epsilon b_\theta,$$

and $j_\theta \sim \epsilon j_\phi$

Use the toroidal component. Since $\mathbf{J} \cdot \nabla \mathbf{B} \ll \mathbf{B} \cdot \nabla \mathbf{J}$, the equation governing the outer region is

$$\mathbf{B} \cdot \nabla j_\phi + \mathbf{b} \cdot \nabla J_\phi = 0$$
Representation of perturbed field

Use a vector potential (flux function) to represent the perturbed flux

\[ \mathbf{b} = \nabla \times \mathbf{a} \text{ with } \mathbf{a} = \psi \hat{\phi} : \text{ so that } b_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad b_\theta = \frac{\partial \psi}{\partial r} \]

which can be related (by Ampere’s law) to the perturbed toroidal current density:

\[ \mu_0 \mathbf{j}_\phi = \nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \]

Assuming that the perturbations are double periodic and have the form \( e^{i(m\theta - n\phi)} \)

\[ b_r = -i \frac{m}{r} \psi \]
**Equation governing outer solution**

Inserting $j_\phi$ into the governing equation, and noting that $J_\phi$ only varies in the radial direction so that $\mathbf{b} \cdot \nabla J_\phi = -i \frac{m}{r} \frac{dJ_\phi}{dr} \psi$ gives

$$\frac{F}{\mu_0} \nabla^2 \psi - m \frac{dJ_\phi}{r \, dr} \psi = 0,$$

where $F(r) = \mathbf{k} \cdot \mathbf{B} = \frac{m}{r} B_\theta(r) - \frac{n B_\phi}{R} = \frac{m}{r} B_\theta(r) \left(1 - \frac{nq(r)}{m}\right)$.

Equivalently,

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = \frac{m^2}{r^2} \psi + \frac{m \mu_0}{r F} \frac{dJ_\phi}{dr} \psi$$

This equation is well-behaved and solvable, except where $F = 0$ (resonance).
Independently solve $\psi$ in two regions: 
I) $0 < r < r_s$ and II) $r_s < r < a$
where $r_s$ is the resonant surface where $F(r_s) = 0$. One way to do this is by by integrating the coupled first order equations from boundaries into resonant surface:

\[
\frac{d}{dr}\psi' = \frac{m^2}{r^2}\psi - \frac{\psi'}{r} + \frac{m\mu_0}{rF} \frac{dJ_\phi}{dr}\psi \\
\frac{d}{dr}\psi = \psi'
\]

Matching conditions: $b_r (\psi)$ continuous, but $\frac{d b_r}{dr} (\psi')$ is discontinuous. Matching to inner solution will determine growth rate. This discontinuity is quantified by

\[
\Delta' = \lim_{\epsilon \to 0} \left[ \frac{\psi'}{\psi} \bigg|_{r_s+\epsilon} - \frac{\psi'}{\psi} \bigg|_{r_s-\epsilon} \right]
\]
HOMEWORK: Numerically calculate $\Delta'$ for tokamak-like profiles of $q \equiv \frac{r B_\phi}{R B_\theta}$ (using Excel, IDL or your favorite programming language). Begin by assuming a current profile of the form
\[ j_\phi = j_0 \left( 1 - \frac{r^2}{a^2} \right) ^\nu \]
where $\nu$ can be related to the safety factor at the axis $q_0$ and at the boundary $q_a$: $\nu = \frac{q_a}{q_0} - 1$. Typically, $q_0 \sim 1$ and $q_a \sim 3 - 5$. For this current profile,
\[ B_\theta(r) = \frac{\mu_0 j_0 a^2}{2r (\nu + 1)} \left( 1 - \left( 1 - \frac{r^2}{a^2} \right) ^{\nu - 1} \right). \]

For these profiles, begin by computing defining a radial grid array $r_i = i \Delta r$ that covers the range $0 \leq r \leq a$ with several hundred discrete radii. Then compute the profiles of $F(r) = k \cdot B$ for an $m = 2, n = 1$ tearing mode. Plot it, and then determine the resonant surface $r_s$ where $F(r_s) = 0$. Next, solve for the perturbed flux in the two regions I) $0 < r < r_s$ and II) $r_s < r < a$. Do this by by integrating the coupled first order equations
\[ \psi_{i+1}' = \psi_i' + \Delta r \left( \frac{m^2}{r_i^2} \psi_i' - \frac{\psi_i'}{r_i} + m \mu_0 \left. \frac{dJ_\phi}{dr} \right|_i \psi_i \right) \]
\[ \psi_{i+1} = \psi_i + \Delta r \psi_i' \]
from boundaries into the resonant surface. For the inside solution, carry out this integration by starting the couple equations with the boundary conditions $\psi(r = 0) = 0$ (regularity), $\psi'(r = 0) = C'$ and integrate outward to $r = r_s - \epsilon$. Use a similar scheme to integrate the equations backwards from the outer wall at $r = a$ to $r = r_s + \epsilon$ (assuming that $\psi(r = a) = 0$ due to a conducting wall). Finally, take your two solutions for the inside and outside regions and scale them to match amplitudes on each side of the resonant surface.

Finally, from the scaled solutions, find the derivatives on either side of the resonance to compute
\[ \Delta' = \lim_{\epsilon \to 0} \left[ \left. \frac{\psi'}{\psi} \right|_{r_s+\epsilon} - \left. \frac{\psi'}{\psi} \right|_{r_s-\epsilon} \right]. \]
Solution

\[ r_s \Delta' = 2.8 \]

B=1 T  
R=1.5 m  
a=0.5 m  
m=2, n=1  
q_a=3.5  
q_0=1.0
Perturbed flux function
Resistive Layer

In the resistive layer, ideal MHD breaks down and two key assumptions fail: the neglect of resistivity in Ohm’s law and and the neglect of inertia in the force balance equation.

Combining Ohm’s law,

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \mu_0 \mathbf{J} \]

with Faraday’s law and Ampere’s law gives the magnetic diffusion equation:

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} - \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} \]
Take the radial component of this equation to give

\[
\frac{\partial b_r}{\partial t} = B \cdot \nabla v_r + \frac{\eta}{\mu_0} \nabla^2 b_r
\]

Assuming that the instability time dependence \(e^{\gamma t}\) and that the growth rate \(\gamma\) is real, \(b_r = -im\psi/r\) and that \(r\) is constant in the layer, gives

\[
\gamma\psi + B_\theta (1 - \frac{nq}{m}) v_r = \frac{\eta}{\mu_0} \nabla^2 \psi
\]

The resistive layer is so narrow that radial derivatives in \(\nabla^2\) will dominate, so that

\[
\frac{d^2 \psi}{dr^2} = \frac{\mu_0 \gamma}{\eta} \psi + \frac{\mu_0 B_\theta}{\eta} \left(1 - \frac{nq}{m}\right) v_r
\]
Assuming $b_r$ does not vary much (constant $\psi$), integrating across layer gives jump in $\Delta'$:

$$
\Delta_{\text{inner}}' = \frac{1}{\psi(r_s)} \left[ \frac{d\psi}{dr} \bigg|_{r_s+\epsilon} - \frac{d\psi}{dr} \bigg|_{r_s-\epsilon} \right]
$$

$$
= \frac{\mu_0 \gamma}{\eta} \int \left( 1 + \frac{B_\theta q'}{\gamma q} (1 - \frac{nq(r)}{m}) \frac{v_r}{\psi(r_s)} \right) dr
$$

Furthermore, we can make a linear expansion of $F$ about the resonance surface so that

$$
1 - \frac{q(r)n}{m} = \frac{q'}{q} \bigg|_{r_s} (r - r_s)
$$

so that

$$
\Delta' = \frac{\mu_0 \gamma}{\eta} \int \left( 1 + \frac{B_\theta q'}{\gamma q} (r - r_s) \frac{v_r}{\psi(r_s)} \right) dr
$$
Equation of motion

In the vicinity of the island, inertia must be taken into account to find $v_r$. The $\phi$ component of the inertial term is

$$\hat{\phi} \cdot \nabla \times \rho \frac{dv}{dt} = \gamma \rho \frac{\partial v_\theta}{\partial r}$$

$$= \gamma \rho \frac{ir \ \partial^2 v_r}{m \ \partial r^2}$$

where incompressibility has been used to relate $v_r$ and $v_\theta$. Including this inertial term in the force balance equation yields

$$\frac{\gamma \rho r^2}{m^2} \frac{d^2 v_r}{dr^2} = \frac{B_\theta}{\mu_0} \left( 1 - \frac{nq(r)}{m} \right) \frac{d^2 \psi}{dr^2} - \frac{dJ_\phi}{d\psi}$$
Using the result from Ohm’s law that relates \(\frac{d^2 \psi}{dr^2}\) to \(v_r\),

\[
\frac{d^2 v_r}{dr^2} - \left(\frac{B_\theta^2 m^2 q'^2}{\rho \gamma \eta r^2 q^2}\right) (r - r_s)^2 v_r = - \frac{B_\theta m^2 q'}{\rho \eta r^2 q} (r - r_s) \psi - \frac{m^2}{\rho \gamma r^2} \frac{dJ_\phi}{dr} \psi
\]

Dimensional analysis gives a characteristic scale length

\[
\delta = \left(\frac{\rho \eta \gamma r_s^2 q^2}{B_\theta^2 m^2 q'^2}\right)^{1/4}
\]

\[
\delta = \frac{(\gamma \tau_R)^{1/4}}{S^{1/2}} \left(\frac{r_s^2 L_q^2}{\alpha^4}\right)^{1/4} a
\]

\[
\delta \sim \frac{(\gamma \tau_R)^{1/4} a}{S^{1/2}}
\]
The equation of motion for $v_r/\psi$ can be recast as

$$\frac{d^2y}{dy^2} - x^2y = -x - C,$$

where $x = (r - r_s)/\delta$, $y = -\frac{\rho \gamma r_s^2 q}{m^2 B_0 q' \delta^3} \frac{v_r}{\psi}$ and

$$a \Delta'_inner = \gamma \tau_R \frac{\delta}{a} \int_{-\infty}^{+\infty} (1 - xy) dx$$

Notes: Only odd part of $y$ contributes to integral; $y'' = -x(1 - xy)$ can be solved numerically to find $y$. Solution for $y$ can be integrated to find

$$a \Delta'_inner = 2.12 \gamma \tau_R \frac{\delta}{a}$$
Expressing $\delta$ in terms of $\gamma \tau_R$ results in

$$a\Delta'_\text{inner} = 2.12 \left( \frac{\gamma \tau_R}{S^{1/2}} \right)^{5/4} \left( \frac{r_s^2 L_q^2}{a^4} \right)^{1/4},$$

or equivalently

$$\gamma \tau_R = 0.55 S^{2/5} \left( \frac{a^4}{r_s^2 L_q^2} \right)^{1/5} (a\Delta'_\text{outer})^{4/5}.$$

Express $\delta$ as

$$\frac{\delta}{a} = 0.86 \frac{(a\Delta'_\text{outer})^{1/5}}{S^{2/5}} \left( \frac{a^4}{r_s^2 L_q^2} \right)^{1/20}$$
Summary

• Finite resistivity allows topology to change, and islands to form

• Outer solution determines stability

\[ \Delta' > 0 \text{ Unstable} \]

• Growth rate is a hybrid between resistive and Alfvén growth rates:

\[ \gamma \sim \frac{(a\Delta'_{\text{outer}})^{4/5}}{\tau_R^{3/5} \tau_A^{2/5}} \]