Lecture # 3. Introduction to Kink Modes – the Kruskal-Shafranov Limit.

Steve Cowley UCLA.

This lecture is meant to introduce the simplest ideas about kink modes. It would take many lectures to develop the theory in all its detail, but hopefully the key ideas can be communicated in 1 hour. We will consider a very simple zero pressure cylindrical equilibrium with nearly constant current in the z direction. When the current is ramped up the plasma becomes unstable to a helical kinking of the plasma column. The dynamics of this instability is our topic. **These notes provide considerable detail – a lot of this detail is impossible to convey in the lecture but I hope you will read the notes at your leisure.** A good exposition of kink modes is given in J.P. Freiberg *Ideal Magneto-Hydrodynamics* Plenum (1987). and in H. Goedbloed and S. Poedts *Principles of Magneto-hydrodynamics* Cambridge (2004).

1 Equilibrium

We consider an equilibrium with constant current in the z direction – $J_z = J_0$ =constant for $r < a$. The basic expansion parameter is *"inverse aspect ratio"* , $\epsilon = a/R_0 \ll 1$. The magnetic field is given by:

$$\boldsymbol{B}_0 = (B_0 + B_2(r))\hat{\mathbf{z}} + \mathbf{B_p}(r) \ \ \textit{with} \ \ B_2 \sim \epsilon^2 B_0 \ \ \textit{and} \ \ |\mathbf{B_p}| \sim \epsilon B_0 \tag{1}$$

Where B_0 is a constant – the dominant "toroidal field" in a tokamak – and r is the cylindrical radius. This ordering of the equilibrium is sometimes called the "tokamak ordering" since it is roughly true for many tokamak experiments. From $\boldsymbol{\nabla} \times \boldsymbol{B}_0 = \mu_0 \mathbf{J_0}$ we obtain (for $r < a$):

$$\mathbf{B_p} = \frac{\mu_0 J_0}{2}(\hat{\mathbf{z}} \times \mathbf{r}) = \frac{B_0}{qR_0}(\hat{\mathbf{z}} \times \mathbf{r}) = \frac{B_p}{r}(\hat{\mathbf{z}} \times \mathbf{r}). \tag{2}$$

where q is constant (see below) and using the force balance equation

$$\mathbf{J_0} \times \boldsymbol{B}_0 = 0 \tag{3}$$

we obtain to $\mathcal{O}(\epsilon^2)$,

$$B_2 = \frac{B_0}{q^2 R_0^2}(a^2 - r^2). \tag{4}$$

The field lines obey the equation:

$$\frac{d\theta}{dz} = \frac{B_p}{rB_0} = \frac{1}{qR_0} \qquad \textit{or equivalently} \qquad \theta = \frac{z}{qR_0} + \theta_0. \tag{5}$$

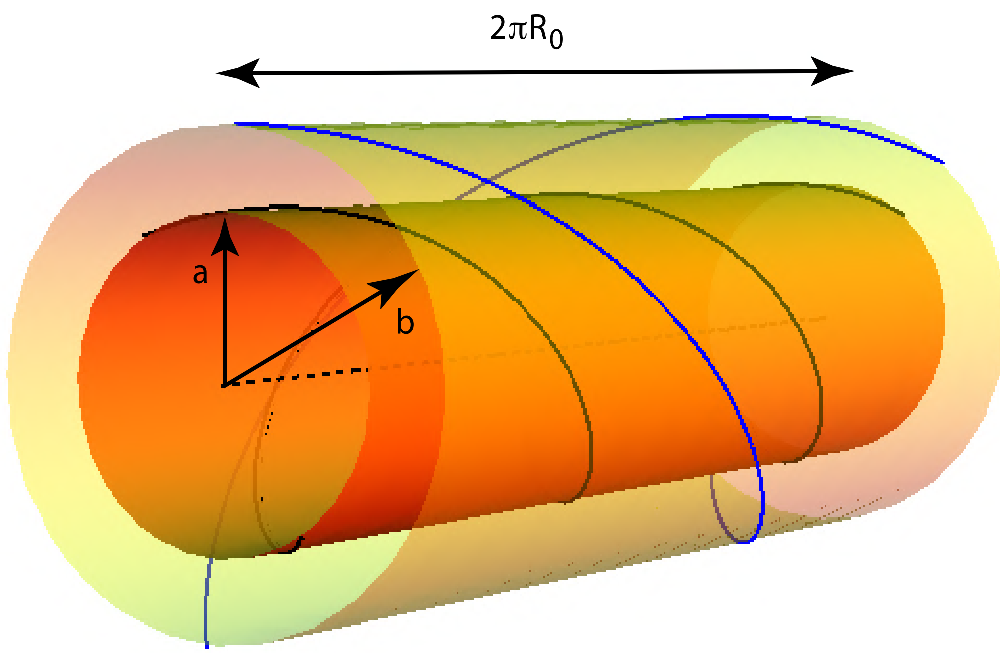

Figure 1: **Cylindrical Screw-pinch** The plasma is contained in the region $r < a$ and the wall at $r = b$ is considered perfectly conducting. The region between $r = a$ and $r = b$ is the *vacuum* region. In the model considered here the (black) field lines in the plasma all have the same *rotational transform*. The blue line is the vacuum field at the wall. The two ends of the pinch are identified – *i.e.* it is periodic in z.

Thus *all* the plasma field lines have the same "twist" – every time we go from one end of the cylinder to the other the plasma field lines go around $1/q$ times in the θ "poloidal" direction. In tokamaks q is called the *"safety factor."* In the Fig. (1) the q is 0.4. Clearly the greater the current the smaller the q – an upper limit on the current is then a lower limit on q.

In the vacuum region the current is, of course, zero and the magnetic field is:

$$\boldsymbol{B}_0 = B_0\hat{\mathbf{z}} + \frac{B_0 a^2}{q r^2 R_0}(\hat{\mathbf{z}} \times \mathbf{r}). \tag{6}$$

The vacuum field has a q profile $q_v(r) = qr^2/a^2$.

2 Plasma Perturbation

To investigate stability we imagine pushing the plasma a small distance away from its equilibrium state. Does the plasma move further away from or return to its equilibrium state? We model the plasma dynamics with Ideal Magneto-hydrodynamics (MHD). The magnetic field obeys: **the frozen-in law**

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \tag{7}$$

which when linearized yields:

$$\delta\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B_0}) = \boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\boldsymbol{B}_0 - \boldsymbol{B}_0\boldsymbol{\nabla} \cdot \boldsymbol{\xi}. \tag{8}$$

where $\boldsymbol{\xi}$ is the displacement of the plasma ($\partial\boldsymbol{\xi}/\partial t = \mathbf{v}$). The perturbed $\mathbf{J} \times \mathbf{B}$ force accelerates the displacement. Thus the **momentum equation** becomes:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{J_0} \times \delta\mathbf{B} + (\boldsymbol{\nabla} \times \delta\mathbf{B}) \times \boldsymbol{B}_0 = \mathbf{F}(\boldsymbol{\xi}). \tag{9}$$

The plasma density ρ_0 is taken to be constant and the pressure is set to zero. If we were doing this formally we would now expand the components of this equation and solve the eigenvalue problem for the growth rate. However we can get to the answer quicker by using our intuition to figure out a good first approximation to the unstable perturbation. We shall look at helical perturbations that move the column. Simply compressing the strong z (toroidal) field makes a stable compressional Alfvén wave. Thus we take the displacement to be dominantly a *rigid helical shift* of the plasma column, *i.e.*

$$\boldsymbol{\xi}_0 = \xi_0 \left[\cos\left(\frac{z}{R_0}\right)\hat{\mathbf{x}} + \sin\left(\frac{z}{R_0}\right)\hat{\mathbf{y}}\right] = \xi_0\hat{\boldsymbol{\xi}}_0. \tag{10}$$

Each poloidal (x, y) plane of the plasma is shifted without distortiona constant distance ξ_0.

The radial displacement is

$$\xi_{r0} = \boldsymbol{\xi}_0 \cdot \hat{\mathbf{r}} = \xi_0 \cos\left(\theta - \frac{z}{R_0}\right) \tag{11}$$

This is the usual way to write the cylindrical displacement. We often expand as helical modes with displacements proportional to $\exp\left(im\theta - in\frac{z}{R_0}\right)$ – here (clearly) we are examining $m = n = 1$ modes.

The field and current perturbations from the rigid displacement are from Eq. (14), using Eqs. (2) and (4):

$$\delta\mathbf{B_0} = \frac{B_0}{qR_0}\left[(1-q)(\boldsymbol{\xi}_0 \times \hat{\mathbf{z}}) + \frac{2\boldsymbol{\xi}_0 \cdot \mathbf{r}}{qR_0}\hat{\mathbf{z}}\right] \quad \delta\mathbf{J_0} = \frac{B_0}{\mu_0 q^2 R_0^2}\left[(2 - q + q^2)(\boldsymbol{\xi}_0 \times \hat{\mathbf{z}})\right] \tag{12}$$

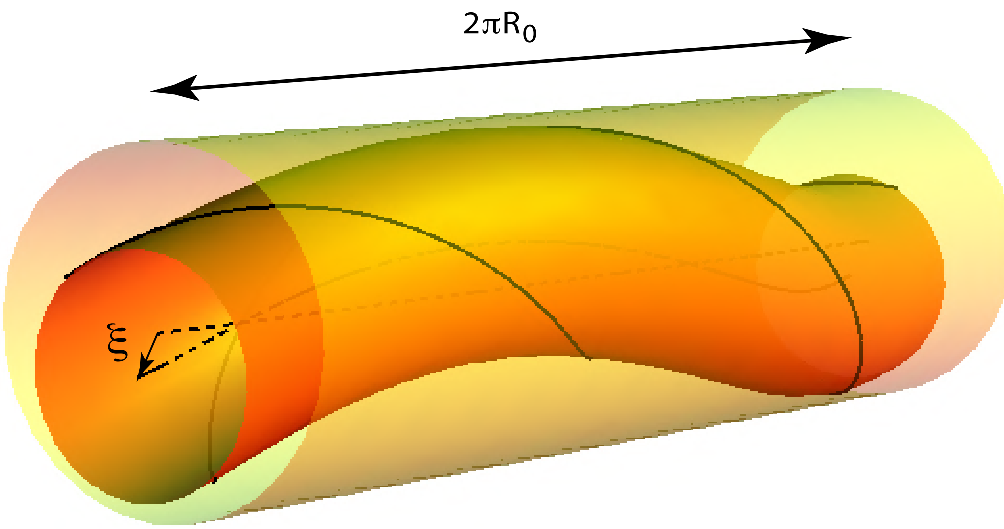

Figure 2: **Helical m=1 n=1 Kink displacement** Rigid displacement of each poloidal plane of the plasma by displacement of the form given by Eq. (10).

The first term in he field perturbation comes from bending the z (toroidal) field. The second term is from moving the varying part toroidal field (B_2). Thus the perturbed force from this displacement is (using Eq. (9)) is given by $\mathbf{F_0} = -\frac{B_0^2}{\mu_0 q R_0^2}(1+q)\boldsymbol{\xi}_0$ – this is a *stabilizing force* since it opposes the motion! This displacement is not quite right. Outside the plasma the vacuum field is perturbed and produces a pressure on the outside of the plasma – this pressure will vary in θ in the same way as the perturbation. We must match the magnetic pressure just inside the plasma with the magnetic pressure just outside the plasma by slightly compressing the plasma. Thus we need to add to our plasma displacement a small compressive component whose magnitude will be determined by the matching of pressures at the boundary. For this compressive perturbation to produce a constant force on the plasma and match the pressure from the vacuum at the boundary it must be quadratic in $\mathbf{r}$ and point in the direction of $\boldsymbol{\xi}_0$. Thus we set:

$$\boldsymbol{\xi}_2 = \alpha \frac{r^2}{2qR_0^2}\boldsymbol{\xi}_0. \tag{13}$$

We shall determine the constant α so that the magnetic pressures balance at the boundary. α is $\mathcal{O}(1)$ so that the compressional displacement is small, $\boldsymbol{\xi}_2 \sim \mathcal{O}(\epsilon^2)\boldsymbol{\xi}_0$. With this small compressional displacement the perturbed field and current become:

$$\delta\mathbf{B} = \frac{B_0}{qR_0}\left[(1-q)(\boldsymbol{\xi}_0 \times \hat{\mathbf{z}}) + \frac{(2-\alpha q)\boldsymbol{\xi}_0 \cdot \mathbf{r}}{qR_0}\hat{\mathbf{z}}\right] \qquad \delta\mathbf{J} = \frac{B_0}{\mu_0 q^2 R_0^2}\left[(2-q\alpha-q+q^2)(\boldsymbol{\xi}_0 \times \hat{\mathbf{z}})\right] \tag{14}$$

We keep the perturbed z magnetic field to $\mathcal{O}(\epsilon^2)$ and the perturbed poloidal field to $\mathcal{O}(\epsilon)$ since they contribute to the magnetic pressure and the forces at the same order. The compression contributes through compressing the z (toroidal) field. After a little algebra the momentum equation becomes:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}_\mathbf{0}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}) = -\frac{B_0^2}{\mu_0 q R_0^2}(1+q-\alpha)\boldsymbol{\xi}_0. \tag{15}$$

3 Vacuum Perturbation

In the vacuum $\boldsymbol{\nabla} \times \delta\mathbf{B} = 0$ so we set $\delta\mathbf{B} = \boldsymbol{\nabla}\chi$. From $\boldsymbol{\nabla} \cdot \delta\mathbf{B} = 0$ we obtain $\nabla^2\chi = 0$. Taking $\chi = \hat{\chi}(r)\sin(\theta - \frac{z}{R_0})$ we find:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\hat{\chi}}{dr}\right) - \frac{\hat{\chi}}{r^2} - \frac{\hat{\chi}}{R_0^2} = 0 \tag{16}$$

We can drop the third term on the left hand side of this equation to lowest order and obtain:

$$\chi = (C_1 r + \frac{C_2}{r} + \mathcal{O}(\epsilon^2))\sin(\theta - \frac{z}{R_0}) \tag{17}$$

Where C_1 and C_2 are constants to be determined. The perturbed vacuum field is:

$$\delta\mathbf{B_v} = (C_1 - \frac{C_2}{r^2})\sin(\theta - \frac{z}{R_0})\hat{\mathbf{r}} + (C_1 r + \frac{C_2}{r})\cos(\theta - \frac{z}{R_0})[\frac{1}{r}\hat{\mathbf{z}} \times \hat{\mathbf{r}} - \frac{1}{R_0}\hat{\mathbf{z}}] \tag{18}$$

At the conducting wall ($r = b$) we can have no radial field thus $C_2 = b^2 C_1$.

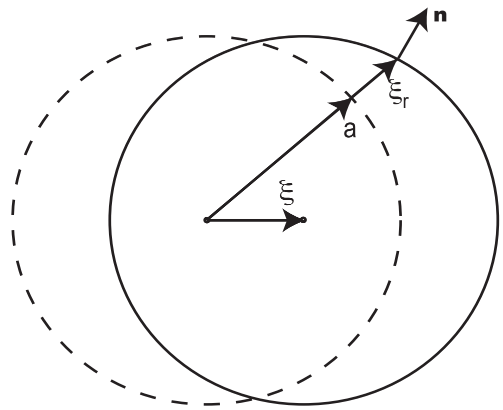

Figure 3: **Vacuum-Plasma Interface.** The shift of the boundary by the displacement – to the order we keep the displacement is just $\boldsymbol{\xi}_0$. The boundary is given by the line $r = a + \xi_r$ and the unit vector normal to the boundary is $\mathbf{n}$.

4 Matching at the Vacuum-Plasma Interface

The normal to the perturbed "flux" surfaces (labeled by their original radius r_0) is in the direction of the $\boldsymbol{\nabla} r_0 = \boldsymbol{\nabla}(r - \xi_r)$. Then to the order we need:

$$\mathbf{n} = \hat{\mathbf{r}} - \boldsymbol{\nabla}\xi_r \tag{19}$$

To join the plasma and vacuum solutions we must satisfy two conditions: 1) no field through the vacuum-plasma interface and, 2) magnetic pressure continuity across the vacuum-plasma interface.

No Vacuum Field Through the Vacuum-Plasma Interface.

$$(\boldsymbol{B}_0 + \delta\mathbf{B}_\mathbf{v}) \cdot (\hat{\mathbf{r}} - \boldsymbol{\nabla}\xi_r) = 0 \quad \rightarrow \quad \boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\xi_r = \delta\mathbf{B}_\mathbf{v} \cdot \hat{\mathbf{r}} \tag{20}$$

Substituting into this expression we determine C_1 in terms of ξ_0:

$$C_1 = \frac{\xi_0 B_0}{R_0} \frac{(1 - \frac{1}{q})}{(1 - \frac{b^2}{a^2})} \tag{21}$$

Magnetic Pressure Continuity Across the Vacuum-Plasma Interface.

Force balance across the plasma boundary becomes the continuity of magnetic pressure.

Thus on the surface $r = r_0 + \xi_r$ we have:

$$B^2_{plasma} = B^2_{vacuum} \tag{22}$$

Using Eq. (14) and a little algebra we find:

$$B^2_{plasma} = B_0^2 + \frac{a^2 B_0^2}{q^2 R_0^2} + \frac{2B_0^2\xi_r}{qR_0^2}(1-\alpha) \tag{23}$$

and from Eqs. (18) and (21) we obtain:

$$B^2_{vacuum} = B_0^2 + \frac{a^2 B_0^2}{q^2 R_0^2} - \frac{2B_0^2\xi_r}{qR_0^2}\left[\frac{1}{q} - q(1-\frac{1}{q})^2\frac{(\frac{b^2}{a^2}+1)}{(\frac{b^2}{a^2}-1)}\right] \tag{24}$$

Equating these expressions we obtain $\alpha = [1 + \frac{1}{q} - q(1-\frac{1}{q})^2\frac{(\frac{b^2}{a^2}+1)}{(\frac{b^2}{a^2}-1)}]$ and substituting into Eq. (15) we obtain:

$$\rho_0\frac{\partial^2\boldsymbol{\xi_0}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}) = \frac{2B_0^2}{\mu_0 q R_0^2}\frac{(\frac{b^2}{a^2}-\frac{1}{q})}{(\frac{b^2}{a^2}-1)}(\frac{1}{q}-1)\boldsymbol{\xi}_0. \tag{25}$$

Clearly we have a growing mode when $1 > q > a^2/b^2$ with growth rate,

$$\gamma = \sqrt{\frac{2B_0^2}{\mu_0\rho_0 q R_0^2}\frac{(\frac{b^2}{a^2}-\frac{1}{q})}{(\frac{b^2}{a^2}-1)}(\frac{1}{q}-1)}. \tag{26}$$

When we set the conducting wall to go to infinity we get the *Kruskal-Shafranov* limit for stability:

$$q > 1 \quad \rightarrow \quad Stable. \tag{27}$$

This is a limit on the current, specifically we must have $\mu_0 J_0 < 2B_0/R_0$ to be stable. Although the stability criterion is very simple all the regions and forces play a role in this instability. Specifically note that the field line bending forces in the plasma – the $1+q$ part of the right hand side of Eq. (15) – are stabilizing. The vacuum pressure is destabilizing since where $\xi_r > 0$ the magnetic pressure on the boundary has decreased.

This arises because the vacuum field decreases with distance from the axis (as $1/r$). The decrease in pressure on the boundary is balanced in the plasma by a reduced z field caused by an expansion of the plasma (where the radial displacement is positive $\xi_r > 0$) – this corresponds to ($\alpha > 0$). The magnetic pressure forces from the expansion/compression inside the plasma are then destabilizing. When $q < 1$ these pressure forces exceed the field line bending forces and the plasma is unstable. Thus this is an *external kink* releasing energy in the vacuum field – indeed the vacuum field lines straighten.

When the wall is close there is a narrow region of instability which corresponds to having the $q = 1$ radius in the vacuum region – *i.e.* $q_v(r_1) = 1 = qr_1^2/a^2$. When the wall is present the flux trapped between the plasma and the wall cannot escape and makes the vacuum field pressure less destabilizing.

5 Other Kink Modes

In Tokamaks kink modes are driven by both current and pressure (Professor Brennan will describe the situation in some detail) and the plasma is not cylindrical. Although the $m = n = 1$ is often the most visible mode number the poloidal (m) harmonics are coupled and none of the modes are a pure poloidal harmonic. The external kinks that are observed can be predominantly higher harmonics. Often what one sees is a mode with $m/n \sim q_{edge}$ – where q_{edge} is the value of q at the edge of the plasma. It is instructive to draw the kinked boundary for the $m = 2$ mode.

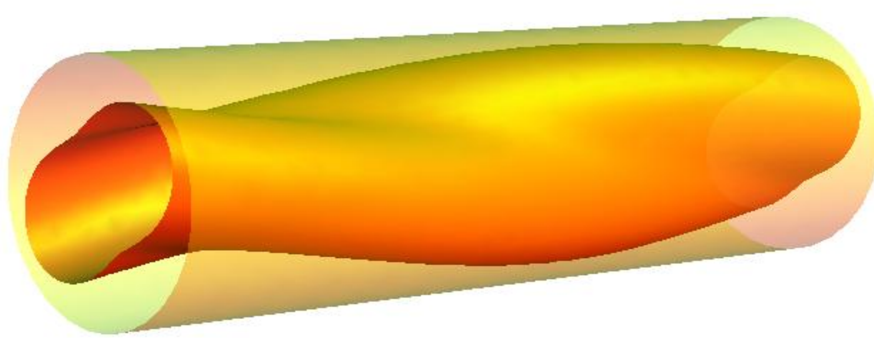

Figure 4: **Helical m=2 n=1 Kink displacement** Eliptical displacement of each flux surface in the plasma by displacement of the form given by $\xi_r \propto \cos(2\theta - \frac{z}{R_0})$.

6 Problem

Vertical and Horizontal Stability. Tokamaks have to be stable to moving up/down or side to side. This puts constraints on the coil arrangement next to the plasma. Lets look at the stability of current filaments as a model of this situation. Consider 5 current filaments (current flowing in the z direction):

- **"Plasma" Filament** initially at $x = y = 0$ representing the plasma carrying current I_{plasma}
- **2 Horizontal "Coils"** at $x = \pm 1,\ y = 0$ carrying current I_0
- **2 Vertical "Coils"** at $y = \pm 1,\ x = 0$ carrying current $-I_0$

Let us imagine that the plasma and coil currents are fixed – i.e. they do not change during any motion of the plasma.

1. Draw the configuration.
2. Calculate the force on the undisplaced plasma?
3. Calculate the force on the "plasma" when it is displaced by a distance δ horizontally.
4. Calculate the force on the "plasma" when it is displaced by a distance δ vertically.
5. In which direction is it unstable? What is the growth rate if the plasma has a mass M?
6. How could you stabilize this instability? Discuss!