Ripples, Twisters and Avalanches in Plasma Phase Space (Nonlinear Physics of Kinetic Instabilities)

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Outline

- Experimental challenges to nonlinear kinetic theory:
  - Pitchfork splitting effect
  - Modes with time-dependent frequencies
  - Bursts in collective losses of fast ions

- Theoretical tool: near-threshold analysis

- Nonlinear phenomena near instability threshold:
  - Ripples (Bifurcations of nonlinearily saturated modes)
  - Twisters (Spontaneous formation of phase-space holes and clumps, associated with strong frequency chirping)
  - Avalanches (Intermittent global diffusion from discrete set of unstable modes)
Nonlinear Splitting of Alfvén Eigenmodes in JET
Rapid Frequency Chirping Events

Hot electron interchange modes in Terrella
(Courtesy of Michael Maul, Columbia University)

Alfvén modes in MAST
(Courtesy of Mikhail Gryaznevich, Culham laboratory, UKAEA)

The ms timescale of these events is much shorter than the energy confinement time in the plasma.
Saturation of the neutron signal reflects anomalous losses of the injected beams. The losses result from Alfvénic activity.

Near-threshold Nonlinear Regimes

• Why study the nonlinear response near the threshold?
  — Typically, macroscopic plasma parameters evolve slowly compared to the instability growth time scale
  — Perturbation technique is adequate near the instability threshold

• Single-mode case:
  — Identification of the soft and hard nonlinear regimes is crucial to determining whether an unstable system will remain at marginal stability
  — Bifurcations at single-mode saturation can be analyzed
  — The formation of long-lived coherent nonlinear structure is possible

• Multi-mode case:
  — Multi-mode scenarios with marginal stability (and possibly transport barriers) are interesting
  — Resonance overlap can trigger hard nonlinear regime
Key Element in Theory

Interaction of energetic particles with unstable waves

- Pendulum equation for particles in an electrostatic wave:
  \[ m \ddot{x} = eE \cos(kx - \omega t) \]
- Wave-particle resonance condition: \( \omega - kv = 0 \)
- Phase space portrait in the wave frame:
Basic Ingredients

- Particle injection and effective collisions, $v_{\text{eff}}$, create an inverted distribution of energetic particles.

- Discrete spectrum of unstable modes.

- Instability drive, $\gamma_L$, due to particle-wave resonance.

- Background dissipation rate, $\gamma_d$, determines the critical gradient for the instability.

\[
\text{Critical slope} \quad \gamma_L = \gamma_d
\]
Joint European Tokamak (JET)
Particle Orbits and Resonances

- Unperturbed particle motion preserves three quantities:
  - Toroidal angular momentum ($P_\varphi$)
  - Energy ($E$)
  - Magnetic moment ($\mu$)

- Unperturbed motion is periodic in three angles and it is characterized by three frequencies:
  - Toroidal angle ($\varphi$) and toroidal transit frequency ($\omega_\varphi$)
  - Poloidal angle ($\theta$) and poloidal transit frequency ($\omega_\theta$)
  - Gyroangle ($\psi$) and gyrofrequency ($\omega_\psi$)

- Wave-particle resonance condition:

\[
\omega - n\omega_\varphi \left( \mu; P_\varphi; E \right) - l\omega_\theta \left( \mu; P_\varphi; E \right) - s\omega_\psi \left( \mu; P_\varphi; E \right) = 0
\]

The quantities $n$, $l$, and $s$ are integers with $s = 0$ for low-frequency modes.
Wave-Particle Lagrangian

• Perturbed guiding center Lagrangian:

\[ L = \sum_{\text{particles}} \left[ P_\theta \dot{\varphi} + P_\varphi \dot{\vartheta} - H \left( P_\theta; P_\varphi; \mu \right) \right] + \sum_{\text{modes}} \delta A^2 \]

\[ + 2 \text{Re} \sum_{\text{particles}} \sum_{\text{modes}} \sum_{\text{sidebands} l} AV_l \left( P_\theta; P_\varphi; \mu \right) \exp \left( -i\alpha - i\omega t + in\varphi + il\vartheta \right) \]

• Dynamical variables:
  • \( P_\theta, \vartheta, P_\varphi, \varphi \) are the action-angle variables for the particle unperturbed motion
  • \( A \) is the mode amplitude
  • \( \alpha \) is the mode phase

• Matrix element \( V_l \left( P_\theta; P_\varphi; \mu \right) \) is a given function, determined by the linear mode structure

• Mode energy: \( W = \omega A^2 \)
Theoretical Formalism

- Unperturbed particle motion is integrable and has canonical action-angle variables \( I_i \) and \( \xi_i \).

- Unperturbed motion is periodic in angles \( \xi_1, \xi_2, \) and \( \xi_3 \).

- Single resonance approximation for the Hamiltonian

\[
H = H_0(I) + 2 \text{Re} [A(t)V(I) \exp(i\xi - i\omega t)]
\]

- Kinetic equation with collisions included

\[
\frac{df}{dt} + \Omega(I) \frac{df}{d\xi} - 2 \text{Re}[iA(t) \exp(i\xi - i\omega t)] \frac{df}{dI} = v^3 \left( \frac{\partial \Omega}{\partial I} \right)^2 \frac{\partial^2 f}{dI^2}
\]

- Equation for the mode amplitude

\[
\frac{dA}{dt} = -\gamma dA + \frac{i\omega}{G} \int d\Gamma V^* \exp(-i\xi + i\omega t)f
\]
Wave Evolution Equation

- Near the instability threshold \((\gamma_L - \gamma_d << \gamma_L)\), the wave saturation time is shorter than the trapped particle bounce period

- Particle response to the wave can be calculated with perturbation technique

- Equation for the normalized wave amplitude \(A\) reduces to

\[
e^{-i\phi} \frac{dA}{dt} = \frac{\gamma}{\cos\phi} A - \frac{\gamma}{2} \int_{t/2}^{t-\tau} \int_{0}^{\tau} \exp[-v_{eff}^3 \tau^2 (2\tau / 3 + \tau_1)] \times A(t - \tau)A(t - \tau - \tau_1) A^*(t - 2\tau - \tau_1)
\]

- The amplitude \(A\) measures the square of the nonlinear bounce frequency, \(\omega_b\), for a typical resonant particle.

- Parameter \(\phi\) characterizes linear properties of the wave.
Steady State Solution with a Source

Continuous source supplies power needed to balance background dissipation
Steady Nonlinear State

- Wave mixes resonant particles and tends to flatten their distribution function.

- Particle source feeds resonant region and maintains a finite slope, $\partial f / \partial \nu$, of the distribution function.

- Nonlinearly reduced growth rate balances the damping rate.

**IS THIS SOLUTION STABLE?**

- Yes, if $\nu_{\text{eff}} > \gamma_L - \gamma_d$

- No, if $\nu_{\text{eff}} < \gamma_L - \gamma_d$
Transition from Steady State Saturation to the Explosive Nonlinear Regime

(1)-saturated mode

(2)-limit cycle

(3)-chaotic nonlinear state

(4)-explosive growth

Instability drive increases from (1) to (4)
Theoretical Fit of the Pitchfork Splitting Experiment

Saturated mode  • First bifurcation  • Period doubling

Time evolution of the bifurcating mode

Simulation

Experiment

- central spectral line
- upshifted sideband
- downshifted sideband

JET Shot #40328     Mode number 8
Onset of Frequency Chirping

- In the limit of low collisionality and strong drive, $\nu_{\text{eff}} \leq \gamma$, the mode follows a self-similar explosive solution:

$$A = \frac{\rho}{(t_0 - t)^{5/2}} \exp [i\sigma \ln(t_0 - t)]$$

- The mode amplitude oscillates at increasing frequency, which provides a seed for further frequency chirping.

- The explosive growth has to saturate at $\omega_b \approx \gamma_L$. 
Mode Evolution Beyond Explosion

Mode lasts many inverse damping rates $\gamma_d$

Mode frequency changes in time
Indication of Coherent Nonlinear Structures

Spatially averaged distribution function

Mode power spectrum
Convective Transport in Phase Space

- Explosive nonlinear dynamics produces coherent structures
  - Convective transport of trapped ("green") particles

- Phase space "holes and clumps" are ubiquitous to near-threshold single mode instabilities
  - Examples: bump-on-tail, TAE’s, etc.

Energy Release via Spontaneous Chirping

- Simulation of near-threshold bump-on-tail instability (N. Petviashvili, 1997) reveals spontaneous formation of phase space structures locked to the chirping frequency.

- Chirp extends the mode lifetime as phase space structures seek lower energy states to compensate wave energy losses due to background dissipation.

- Clumps move to lower energy regions and holes move to higher energy regions.

Interchange of phase space structures releases energy to sustain chirping mode.
Mode Pulsation Scenario

1. Unstable wave grows until it flattens the distribution of resonant particles; the instability saturates when \( \omega_b = \gamma_L \).

2. The excited wave damps at a rate \( \gamma_d < \gamma_L \) with the distribution function remaining flat.

3. The source restores the distribution function at a rate \( v_{\text{eff}} \), bringing a new portion of free energy into the resonance area.

4. The whole cycle repeats.
Recurrent Chirping Events Maintain Marginally Stable Distribution

Relaxation of unstable double-humped distribution, with source, sink, and background plasma dissipation.  

_Recurrence, et al., PRL (2007), to be published_

Instability reduces stored energetic particle energy but does not affect power deposition into the plasma.
Phase Space Structures and Fast Chirping Phenomena

Validation of nonlinear single-mode theory in experiments and simulations

Reproduction of recurring chirping events on MAST

Initial results from multi-mode simulations (Sherwood 2006)

Nonlinear excitation of stable mode

Mode overlap enhances wave energy
Effect of Resonance Overlap

The overlapped resonances release more free energy than the isolated resonances.
What Happens with Many Modes

Benign superposition of isolated saturated modes when resonances do not overlap

Enhanced energy release and global quasilinear diffusion when resonances overlap
Fusion Motivation

• Species of interest: Alpha particles in burning plasmas
  NBI-produced fast ions
  ICRH-produced fast ions
  Others...

• Initial fear: Alfvén eigenmodes (TAEs) with global spatial structure may cause global losses of fast particles

• Second thought: Only resonant particles can be affected by low-amplitude modes
Transport Mechanisms

- **Neoclassical:** Large excursions of resonant particles (banana orbits) + collisional mixing
- **Convective:** Transport of phase-space holes and clumps by modes with frequency chirping
- **Quasilinear:** Phase-space diffusion over a set of overlapped resonances

**Important Issue:** Individual resonances are narrow. How can they affect every particle in phase space?
Intermittent Quasilinear Diffusion

A weak source (with insufficient power to overlap the resonances) is unable to maintain steady quasilinear diffusion.

Bursts occur near the marginally stable case.

\[ f \]

Classical distribution

Metastable distribution

Marginal distribution

Sub-critical distribution
Numerical simulations of Toroidal Alfvén Eigenmode (TAE) bursts with parameters relevant to TFTR experiments have reproduced several important features:

- synchronization of multiple TAEs
- timing of bursts
- stored beam energy saturation

Phase Space Resonances

For low amplitude modes:
\[ \delta B/B = 1.5 \times 10^{-4} \]
\[ n=1, n=2, n=3 \]

At mode saturation:
\[ \delta B/B = 1.5 \times 10^{-2} \]
\[ n=1, n=2, n=3 \]
Temporal Relaxation of Radial Profile

- **Counter-injected beam ions:**
  - Confined only near plasma axis

- **Co-injected beam ions:**
  - Well confined
  - Pressure gradient periodically collapses at criticality
  - Large pressure gradient is sustained toward plasma edge
Issues in Modeling Global Transport

• **Reconciliation of mode saturation levels with experimental data**
  — Simulations reproduce experimental behavior for repetition rate and accumulation level
  — However, saturation amplitudes appear to be larger than the experimental values

• **Edge effects in fast particle transport**
  — Sufficient to suppress modes locally near the edge
  — Need better description of edge plasma parameters

• **Transport barriers for marginally stable profiles**

• **Transport properties of partially overlapped resonances**
Concluding Remarks

- All particles are equal but resonant particles are more equal than others.

- Near-threshold kinetic instabilities in fusion-grade plasmas exhibit rich but comprehensible non-linear dynamics of very basic nature.

- Nonlinear physics offers interesting diagnostic opportunities associated with bifurcations and coherent structures.

- Energetic particle driven turbulence is prone to intermittency that involves avalanche-type bursts in particle transport.