Heat Transport in a Stochastic Magnetic Field

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Magnetic perturbations can destroy the nested-surface topology desired for magnetic confinement.

- Stochastic instability occurs when magnetic islands overlap, causing the field lines to wander randomly throughout the plasma volume.
- Parallel streaming along the stochastic field leads to radial transport.
- Astrophysical plasmas have weak ordered field (naturally “tangled”)

(B perturbations from instability or “error” components)
Projection of radial field yields intuitive estimate of stochastic transport.

Recall parallel heat transport

\[
\frac{\partial T}{\partial t} = \chi_{\parallel}(\hat{b} \cdot \nabla)^2 T \quad \text{where} \quad \hat{b} = \frac{B}{B}
\]

If \( B = B_0 + \tilde{B}_r \hat{r} \) where \( B_0 \) = well-ordered field, forming nested magnetic surfaces

\[
\frac{\partial T}{\partial t} = \chi_{\parallel}(\hat{b} \cdot \nabla)^2 T = \chi_{\parallel} \left( \frac{\tilde{B}_r}{B_0} \right)^2 \frac{\partial^2 T}{\partial r^2}
\]

(not quite rigorous, ok for fluid limit)

effective perpendicular transport
Small fluctuation amplitudes can yield large transport.

Recall for classical electron transport

\[
\frac{\chi_\parallel}{\chi_\perp} \sim \frac{\lambda_{mfp}^2 \nu_c}{\rho^2 \nu_c} \geq 10^6
\]

Small magnetic fluctuation amplitude yields substantial transport

\[
\chi_\parallel \left( \frac{\tilde{B}_r}{B_0} \right)^2 \sim \chi_\perp \quad \text{for} \quad \frac{\tilde{B}_r}{B_0} \sim 10^{-3}
\]
Outline.

- Model for stochastic transport
- Comparisons with experimental measurements (mostly from the RFP)
Fluctuation-induced transport fluxes.

Linearizing the drift kinetic equation

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = 0 \]

\[ f = f_0 + \tilde{f} \]

\[ \frac{\partial f_0}{\partial t} = -\tilde{\mathbf{v}} \cdot \nabla \tilde{f} = -\left( \frac{\tilde{\mathbf{E}} \times \mathbf{B}_0}{B_0^2} + v_{||} \frac{\tilde{\mathbf{B}}}{B_0} \right) \cdot \nabla \tilde{f} \]

- Drift associated with electrostatic fluctuations
- Streaming associated with magnetic fluctuations
Fluctuation-induced transport fluxes.

Moments of the d.k.e. lead to the fluctuation-induced transport fluxes

particle: \( \Gamma_r = \langle \int dv \tilde{f} \left( \frac{\tilde{E} \times B_0}{B_0^2} + v_\parallel \frac{\tilde{B}}{B_0} \right) \cdot \hat{r} \rangle = \langle \tilde{n}\tilde{E}_\perp \rangle / B_0 + \langle \tilde{J}_\parallel \tilde{B}_r \rangle / eB_0 \)

energy: \( Q_r = \langle \int dv v^2 \tilde{f} \left( \frac{\tilde{E} \times B_0}{B_0^2} + v_\parallel \frac{\tilde{B}}{B_0} \right) \cdot \hat{r} \rangle = \langle \tilde{p}\tilde{E}_\perp \rangle / B_0 + \langle \tilde{Q}_\parallel \tilde{B}_r \rangle / B_0 \)

where \( \langle \ldots \rangle \) denotes an appropriate average, e.g., over an unperturbed magnetic flux surface.
Model for stochastic magnetic transport.

- Very few self-consistent models for magnetic fluctuation induced transport have been developed.

- Most analysis has been for a static, imposed set of magnetic fluctuations
  - Error fields from misaligned magnets and other stray fields
  - Low frequency turbulence

- Stochastic magnetic transport is described by a double diffusion process
  1. Random walk of the magnetic field lines
  2. Collisional or other cross-field transport process is required for particles to “lose memory” of which field line they follow
Magnetic diffusion.

Divergence of neighboring field lines:

\[ r(s) = r_0 e^{s/L_K} \]

\[ \delta(s) = r_0 e^{-s/L_K} \]

\( L_K \) = Kolmogorov-Lyaponov length
Magnetic diffusion.

Magnetic diffusion coefficient:

\[
D_m = \frac{\langle (\Delta r)^2 \rangle}{\Delta s} = \frac{\int_0^\infty \tilde{B}_r(0)\tilde{B}_r(s)ds}{B_0^2} \quad \text{(units of length)}
\]

\[
= L_{ac} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2} \quad L_{ac} = \text{auto-correlation length for } \tilde{B}
\]

\(L_{ac}\) is related to the width of the \(k_\parallel\) spectrum, \(L_{ac} \approx \pi / \Delta k_\parallel \quad (=L_K)\) in general.
Consider a test particle streaming along the magnetic field.

Distance, $s$, along unperturbed field $B_0$.

Average radial displacement associated with field line diffusion:

$$\langle (\Delta r)^2 \rangle = D_m \Delta s$$

For $\lambda_{mfp} \gg L_{ac}$,

$$\chi_{st} = \frac{\langle (\Delta r)^2 \rangle}{\Delta t} = \frac{D_m \lambda_{mfp}}{\tau_c} = D_m \nu_T$$

$$\nu_T = \sqrt{T/m}$$ (thermal velocity)

$$\tau_c = \lambda_{mfp} / \nu_T$$ (collision time)
Stochastic transport in the collisional limit.

For $\lambda_{mfp} \ll L_{ac}$, test particle must “first” diffuse $\Delta s \sim L_{ac}$ along the field.

The parallel diffusion is given by:

$$\chi_{||} = \frac{\langle (\Delta s)^2 \rangle}{\Delta t} = \frac{\lambda_{mfp}^2}{\tau_c}$$

$$\chi_{st} = \frac{\langle (\Delta r)^2 \rangle}{\Delta t} = \frac{D_m \Delta s}{\Delta t} = \frac{D_m L_{ac}}{L_{ac}^2 / \chi_{||}} = D_m v_T \left( \frac{\lambda_{mfp}}{L_{ac}} \right) = \chi_{||} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2}$$
Stochastic transport in the collisional limit.

For $\lambda_{mfp} << L_{ac}$, test particle must “first” diffuse $\Delta s \sim L_{ac}$ along the field.

The parallel diffusion is given by:

$$\chi_{\parallel} = \frac{\langle (\Delta s)^2 \rangle}{\Delta t} = \frac{\lambda_{mfp}^2}{\tau_c}$$

$$\chi_{st} = \frac{\langle (\Delta r)^2 \rangle}{\Delta t} = \frac{D_m \Delta s}{\Delta t} = \frac{D_m L_{ac}}{L_{ac}^2 / \chi_{\parallel}} = D_m v_T \left( \frac{\lambda_{mfp}}{L_{ac}} \right) = \chi_{\parallel} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2}$$

Smooth transitional form:

$$\chi_{st} = v_T L_{eff} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2} \quad \text{with} \quad L_{eff}^{-1} = L_{ac}^{-1} + \lambda_{mfp}^{-1}$$

Krommes et al. provided a unifying discussion of various collisional limits with respect to characteristic scale lengths.
How well does the static field model work?

- Few direct measurements of stochastic transport.

- Inferences via energetic particles in tokamak plasmas, exploiting expected velocity dependence.

- Self-organizing plasmas like the RFP and spheromak provide good opportunity to test expectations, because they exhibit a broad spectrum of low frequency magnetic fluctuations.
The Reversed Field Pinch plasma configuration.

MST parameters:
- \( n \sim 10^{13} \text{ cm}^{-3} \)
- \( T_e < 2 \text{ keV} \)
- \( T_{ion} \sim T_e \)
- \( B < 0.5 \text{ T} \)
- \( \rho_{ion} \sim 1 \text{ cm} \)
The MST at UW-Madison.

$R = 1.5\ m$

$\alpha = 0.5\ m$

$I_p < 0.6\ MA$
Main source of symmetry breaking magnetic field in the RFP is MHD tearing instability.

- Linear stability analysis using force balance \( J \times B = \nabla p \) yields

\[
\ddot{\tilde{B}}_r + \left( \frac{F''}{F} + k^2 \right) \tilde{B}_r \approx 0 \quad \text{with} \quad k = \frac{m}{r} \hat{\theta} - \frac{n}{R} \hat{\phi} \quad F = k \cdot B_0
\]

\[
\frac{F''}{F} \approx \frac{B_0''}{B_0} \approx \nabla_r \left( \frac{J_{||}}{B} \right)
\]

- Growth rate depends on \( \nabla_r J_{||} \) and the plasma’s resistivity

- Mode resonance appears at the minor radius where \( k \cdot B_0 \gg 0 \)

\[
k \cdot B_0 = 0 \quad \Rightarrow \quad m = \frac{r B_0 \phi}{nRB_\theta} = q(r)
\]

(see 2008 Winter School lectures)
Tearing permits the creation of magnetic islands.

Resonant layer $k \cdot B_0 \Rightarrow 0$

$r = r_s$

Perpendicular to $B_0$

Tearing reconnection

Magnetic island forms

Island width

$$w_{m,n} = 4 \sqrt{\frac{\tilde{B}_{r,m,n}(r_s)L_s}{B_0(r_s)k_\perp}}$$
Chirikov threshold condition for stochastic instability.

If neighboring magnetic islands overlap, the field lines are allowed to wander from island-to-island randomly.

\[ s = \frac{1}{2} \frac{w_{n+1} + w_n}{r_{s,n+1} - r_{s,n}} \]  

“stochasticity parameter”
(cruely the number of islands overlapping a given radial location)

\( s < 1 \): islands do not overlap, no stochastic transport
(but transport across the island is typically enhanced by its topology)

\( s \sim 1 \): weakly stochastic, magnetic diffusion and transport are transitional
(e.g., as discussed by Boozer and White)

\( s \gg 1 \): magnetic field line wandering is well approximated as a random-walk diffusion process
Many possible tearing resonances occur across the radius of the RFP configuration.

\[ q(r) = \frac{rB_\phi}{RB_\theta} \]

\(~0.2\)

\(m=1, \ n \geq 6\) resonances

Observed Spectrum
Chirikov threshold is exceeded, particularly in the mid-radius region where the density of rational magnetic surfaces is large.
Magnetic puncture plot indicates widespread magnetic stochasticity.

Eigenfunctions from nonlinear resistive MHD computation, normalized to measured $\tilde{B}_{m,n}(r = a)$.

Field is modeled using $\tilde{B}_{m,n}(r)$ eigenfunctions, combined with equilibrium reconstruction that provides $B_0(r)$. 

$$\tilde{B}_{r,m,n}(r)$$
Direct measurement of magnetic fluctuation-induced stochastic transport.

Measurements were made in MST (RFP), CCT (tokamak), and TJ-II (stellarator).
Measured electron heat flux in the edge of MST plasmas.
Measured island-induced heat flux in CCT (tokamak at UCLA).

Heat flux in the magnetic island scales as if stochastic.

\[
\langle \tilde{Q}_\parallel \tilde{B}_r \rangle / B \sim 0.2 \text{ W/cm}^2
\]

\[
\tilde{B}_r / B \sim 7 \times 10^{-4}
\]

\[ q = 2 \]

\[ w_{island} \sim 6-7 \text{ cm} \]

Estimated \[ \chi_{R-R} n \nabla T_e \sim 2.5 \times 10^5 \tilde{B}_r^2 / B_0^2 \]
The amplitude of the tearing fluctuations in the RFP can be reduced using current profile control (PPCD). ~5X reduction of most modes allows tests of 

\[ \chi_{st} \sim \tilde{B}_r^2 \] 

scaling and dependence on spectral features.
Region of stochastic field shrinks with current profile control.
Power balance measurements provide the experimental electron heat conductivity profile.

Electron heat flux

\[ Q_e = \chi_e n \nabla_r T_e \]
Measured heat diffusivity consistent with collisionless stochastic transport model (where the field is stochastic).

Magnetic diffusivity is evaluated directly from an ensemble of magnetic field lines.

\[ L_{ac} \ll \lambda_{mfp} \]

\[ \sim 1 \text{ m} \quad \sim 10\text{'s m} \]

\[ \chi_{st} = D_m v_T \]
Magnetic diffusivity as expressed by Rechester-Rosenbluth, PRL ’78.

\[ D_m = \pi R \sum_{m,n} \frac{|B_{r,m,n}(r)|^2}{B_z^2} \delta \left[ \frac{m}{n} - q(r) \right] \]

- auto-correlation length, \( L_{ac} \)
- RMS fluctuation amplitude^2
- \( \ldots \) but **only** \( k_{\parallel} = 0 \) modes resonant nearby \( r \)
Estimate of the auto-correlation length from the spectral width.

For a tokamak $B_\phi >> B_\theta$

$$k_\parallel = \frac{k \cdot B}{B} \approx \frac{1}{B_\phi} \left( \frac{m}{r} B_\theta + \frac{n}{R} B_\phi \right) = \frac{1}{R} \left( \frac{m}{q} + n \right)$$

$$\Delta k_\parallel \sim \Delta r \underbrace{\left| \frac{\partial k_\parallel}{\partial r} \right|}_r_s = \frac{n}{R} \Delta r \left| \frac{1 dq}{q \, dr} \right|_r_s \sim \frac{1}{R} \quad (n=1 \text{ typically dominant})$$

mode radial width
Rechester-Rosenbluth magnetic diffusivity overestimates $\chi_{st}$ for regions with low $s$. 
Electron temperature gradient correlates with amplitude of tearing modes resonant at mid-radius.

\[ \tilde{B}_{rms} = \sqrt{\sum_{n=8}^{15} \tilde{B}_n^2(a)} \]
Electron temperature gradient does **not** correlate with largest mode, resonant in the core.

$$\tilde{B}_{rms} = \sqrt{\sum_{n=8}^{15} \tilde{B}_n^2(a)}$$
Though parallel streaming transport is nonlocal, the tearing reconnection process is local.

\[ D_m \sim \sum_{m,n} \frac{|B_{r,m,n}(r)|^2}{B_z^2} \delta[m/n - q(r)] \]

illustrates importance of \( k_{\parallel} = 0 \)

RMS \( m = 1, n = 8-15 \)

linear eigenmodes
Stochastic particle transport is affected by its inherent non-ambipolar character.

Since the thermal velocity is mass-dependent, electron and ion stochastic diffusion are not automatically ambipolar (unlike $\vec{E} \times \vec{B}_0$ motion).

Harvey derived from the drift kinetic equation (collisionless limit)

$$\Gamma_r \sim D_m v_T \left( \frac{1}{n} \frac{\partial n}{\partial r} + \frac{1}{2T} \frac{\partial T}{\partial r} + \frac{eE_A}{T} \right) n$$

$$Q_r \sim D_m v_T \left( \frac{1}{n} \frac{\partial n}{\partial r} + \frac{3}{2T} \frac{\partial T}{\partial r} + \frac{eE_A}{T} \right) nT$$

Setting $\Gamma_{r,e} \approx 0$ yields the ambipolar electric field $E_A = -\frac{T_e}{e} \frac{\partial}{\partial r} \ln(nT_e^{1/2})$

and $Q_{r,e} \sim D_m v_T n \frac{\partial T}{\partial r}$
Non-ambipolar transport predicts a radially outward directed electrostatic field due to the high mobility of electrons.

Heavy ion beam probe observes the positive potential in the core.

Lei et al.
In astrophysical plasmas, stochastic field can reduce heat transport.

Reflects large transport anisotropy in a magnetized plasma.

Consider collisionless limit $L_{ac} \ll \lambda_{mfp}$:

$$
\chi_{st} = D_m v_T = \chi_\parallel \frac{D_m}{\lambda_{mfp}} = \chi_\parallel \frac{L_{ac}}{\lambda_{mfp}} \left( \frac{\tilde{B}_r}{B_0} \right)^2 < 1, \text{ even for } \tilde{B} \sim B_0
$$

Has been applied to cooling flows in galactic clusters to argue small heat conduction.
a) Consider the RFP magnetic equilibrium. Using the Chirikov stochasticity parameter, derive from the threshold condition, $s=1$, the recursion relation below for the width of the magnetic island associated with toroidal mode, $n$, so that it just touches its nearest neighbors (assume $m=1$ for all modes):

$$w_n = \frac{1}{|q'_n|} \frac{1}{n(n+1)}$$

where

$$q'_n = \left. \frac{dq}{dr} \right|_{r=r_n}$$

and $r_n$ is the minor radius of the resonant surface.

b) Estimate the stochastic heat diffusivity, $\chi_{st}$, for a fluctuation spectrum described by the recursion relation above.

c) For fusion parameters, discuss the magnitude of $\chi_{st}$ relative to other transport mechanisms, such as classical (or neoclassical) transport and anomalous transport as observed in tokamak plasmas. For what $n$ is $\chi_{e, st} < 1 \text{ m}^2/\text{s}$?
Homework problem (illustration and partial answer)

\[ n = 6 \]

\[ w_n = 4 \frac{\tilde{B}_{r,n}(r_n)r_n}{B_\theta(r_n) n |q_n'|} \]

(a=plasma minor radius)